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Minimal Hyperspheres in Rank Two Compact Symmetric Spaces

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Abstract. We describe a method to construct embedded, minimal hyperspheres in rank two compact symmetric spaces which are equivariant under the isotropy action of the symmetric space, and we supply the details of the construction for the exceptional Lie group G_2.

0. Introduction

The simply-connected spaces of constant curvature can be characterized by the property of being reflectionally symmetric with respect to any given direction (a reflectional symmetry is an involutive isometry which reverses only one direction), i. e. the group of the isometries of the space which fix any given point is the largest possible, namely, equal to $O(n)$. They are the Euclidean spaces, the spherical spaces and the hyperbolic spaces.

The family of symmetric spaces constitutes a natural generalization of the spaces of constant curvature. A Riemannian manifold is called a symmetric space if it is centrally symmetric with respect to any point, that is, the isotropy subgroup contains an involutive isometry which is the geodesic symmetry. Among them, the symmetric spaces of compact (resp., non-compact) type are generalizations of the spherical (resp., hyperbolic) spaces.

It is therefore natural to investigate to what extent fundamental re-

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sults about the classical spaces of constant curvature remain true in the larger family of symmetric spaces. In particular, among all hypersurfaces of $S^n(1)$, the equator appears as the "simplest" global object, e.g. it is the unique closed minimal hypersurface of $S^n(1)$ with least total volume. Moreover, from the differential topological viewpoint, spheres are the most basic closed manifolds. So, in [**7**] W. T. Hsiang, W. Y. Hsiang and P. Tomter conjectured that every simply-connected, compact symmetric space of dimension at least four must contain some embedded, minimal hyperspheres (i.e., hypersurfaces diffeomorphic to a sphere) and they proposed that those minimal hyperspheres should be the candidates to generalized equators in compact symmetric spaces. In the same paper, the method of equivariant geometry was used to establish the existence of such objects in the four compact symmetric spaces of A_2-type. Furthermore, many more examples in other compact symmetric spaces of rank one and two have been constructed (see [**7,8,3**]). The purpose of this paper is to show how the equivariant construction of those generalized equators can be possibly extended to the remaining compact symmetric spaces of rank two. We now explain this in more detail.

Let G/K be an irreducible compact symmetric space. Under the isotropy action of K on G/K, the orbit space $K\backslash G/K$ can be identified isometrically with the closed Cartan polyhedron, which is a flat triangle in the rank two case. The PDE for minimal hypersurfaces in G/K reduces to an ODE in that triangle. The interesting feature of the ODE is that the boundary of the orbit space is comprised of singular points. We search for embedded, minimal hyperspheres in G/K which are K-equivariant. Their generating curves in the orbit space are embedded solutions of some geometric type of the reduced ODE.

The technique that is utilized to find closed solutions to the reduced ODE is based on a comparison lemma (see Section 4) adapted from the one in [**10**]. It enables us to find suitable approximations to solutions of the ODE. We then prove the existence of the desired closed solution by continuity arguments. We have already used this approach in [**4**].

The complete details of the construction are presented for the exceptional Lie group G_2. But the method also provides examples of at least one embedded, minimal hypersphere in each one of the symmetric spaces:

$$\frac{Sp(4)}{Sp(2)\times Sp(2)}, \frac{SU(5)}{S(U(2)\times U(3))}, \frac{Sp(5)}{Sp(2)\times Sp(3)}, \frac{SO(10)}{U(5)},$$
$$\frac{E_6}{U(1)\times_{\mathbb{Z}_2} Spin(10)} \quad \text{and} \quad \frac{G_2}{SO(4)}.$$

In particular, the inverse images of those minimal hyperspheres supply new examples of codimension one closed minimal submanifolds in $Sp(4)$, $SU(5)$, $Sp(5)$, $SO(10)$, E_6 and G_2. The examples in $E_6/(U(1)\times_{\mathbb{Z}_2} Spin(10))$, G_2 and $G_2/SO(4)$ are believed to be the first examples of closed minimal hypersurfaces in those spaces.

This verifies the Hsiang-Hsiang-Tomter conjecture for all symmetric spaces of rank at most two, except for the higher dimensional Grassmannians $SU(2+m)/S(U(2)\times U(m))$ and $Sp(2+m)/(Sp(2)\times Sp(m))$, $m \geq 4$. Since this is a case-by-case verification, they have been left out. The conjecture is certainly true for them, too; what remains to be done is purely some more computational work.

1. Symmetric Spaces

Let $M = G/K$ be a simply connected, compact, irreducible globally symmetric space of dimension n, where G is the connected group of isometries of M and K is the isotropy subgroup of a chosen basepoint o. Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G, K, respectively. The symmetry at o induces an involution of $\mathfrak{g}$; let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding decomposition into the ± 1-eigenspaces. The action of K on M is called the isotropy action of the symmetric space. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. Then $A = \mathrm{Exp}(\mathfrak{a})$ is a maximal flat, totally geodesic submanifold of M which intersects every K-orbit in M orthogonally. Let $\mathfrak{h} = \mathfrak{a} + (\mathfrak{k}\cap\mathfrak{h})$ be a Cartan subalgebra of $\mathfrak{g}$, $\Delta(\mathfrak{g},\mathfrak{a})$ the set of restricted roots, $\Sigma = \{\alpha_1, \cdots, \alpha_p\}$ the simple roots, and β the highest root (with respect to some chosen ordering). Then it follows from É. Cartan's work ([**1,2**]) that the orbit space M/K is isometric to

A/W, where W is the Weyl group of M generated by the reflections on the orthogonal hyperplanes of the restricted roots. In turn, the quotient A/W is isometric to the Cartan polyhedron in $\mathfrak{h}$, defined by the inequalities $\alpha_1 \geq 0, \cdots, \alpha_p \geq 0$, $\beta \leq \pi$.

2. Equivariant Differential Geometry

The reader may want to consult the references [**9,11,12,7**] for information about the results stated below. In what follows, M is a simply-connected, compact symmetric space of rank 2. Then the orbit space X of the isotropy action (K, M) can be identified isometrically with the Cartan polygon which is a flat triangle in $\mathfrak{h} \cong \mathbb{R}^2$. Note that the interior points of X parametrize principal orbits and the boundary points of X parametrize singular orbits.

We intend to construct an embedded minimal hypersphere (i.e., an hypersurface of the diffeomorphic type of an sphere) in M which is equivariant with respect to the isotropy action of K. If N is an equivariant minimal hypersurface in M, it is generated by a curve $\gamma = N/K$ in X, parametrized by arc-length, which is a solution of the ordinary differential equation

$$k(s) = \frac{d}{dn} \log v(\gamma(s)) \tag{1}$$

where k is the geodesic curvature of γ, n is its oriented normal, and $v : X \to \mathbb{R}^+$ is the volume functional which registers the $(n-2)$-dimensional volume of the fibers. Since v vanishes on the singular strata of X, the ODE (1) becomes singular there. Nevertheless, many interesting solutions providing examples of closed submanifolds N originate and terminate at boundary points of X. In fact, a non-self-intersecting solution curve connecting boundary points in adjacent edges of X which meet at a corner which is a fixed point of the K-action generates an embedded minimal *hypersphere* in M.

For a boundary point P which is not a corner, the singularity is of the regular type studied in [7, pp. 587–589] (see also [**13**]), from where we know that there is a unique continuous curve $\gamma(s)$ in X which defines an solution of the reduced minimal equation with $\gamma(0) = P$. Furthermore, γ

is analytic in (P, s) as long as it does not intersect the singular boundary again, and it is perpendicular to the boundary at P.

Any solution curve which hits the boundary can be continued back along the same trajectory with a discontinuous jump in the angle at the boundary. Such a boundary point is called a *bouncing-back* point. Close by solutions will generically avoid the boundary, but we have the phenomenon of "sharp-turning" close to the boundary (see [11, pp. 205–207]). By examining that phenomenon, one can show that a compact segment of a solution of the reduced minimal ODE depends continuously on the initial conditions of the solution, also when the segment contains some bouncing-back points on the singular interior edges, and a compact segment which do not contain bouncing-back points depends in a $\mathcal{C}^1$ way on the initial conditions of the solution.

It is also known that there is a unique solution to the reduced minimal equation emanating from a corner point and it is analytic.

4. The Explicit Form of the Reduced Minimal Equation

The orbit space X is a flat triangle in $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ defined by the following inequalities, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and $\beta \leq \pi$, where α_1, α_2 are the simple roots and β is the highest root.

The volume functional can be computed to be

$$v(x) = c_M \prod_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})} \sin^{m_\alpha} \alpha(x) \qquad (x \in X)$$

where c_M is a constant depending only on M and m_α is the multiplicity of α as a restricted root of M. Write a solution to the reduced minimal ODE as $\gamma(s) = (x(s), y(s))$, where s is the arc-length parameter, and let σ denote the angle from $\partial/\partial x$ to the tangent direction $d\gamma/ds$. Then eqn. (1) can be conveniently expressed as

$$\begin{aligned} \dot{x} &= \cos\sigma \\ \dot{y} &= \sin\sigma \\ \dot{\sigma} &= \sum_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})} m_\alpha \cot(a_\alpha x + b_\alpha y)(b_\alpha \cos\sigma - a_\alpha \sin\sigma) \end{aligned} \tag{2}$$

where the constants a_α, b_α are determined by

$$\alpha(x, y) = a_\alpha x + b_\alpha y$$

for each $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$.

The constants appearing in eqn. (2) (for some of the compact symmetric spaces of rank two which have not been considered in [**8**]) are listed in Tables 1 and 2. In each case, the highest root is indicated with an asterisk. It follows that the orbit space X is as in Figs. 1, 2 and 3.

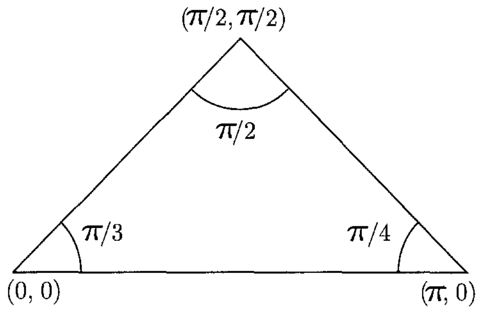

Figure 1 – Orbit space in the case $\frac{Sp(4)}{Sp(2)\times Sp(2)}$

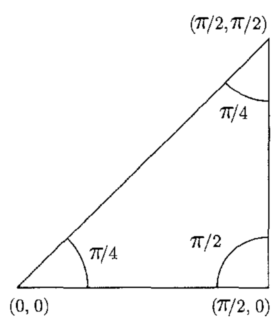

Figure 2 – Orbit space in the case $\circ = \circ$**, except** $\frac{Sp(4)}{Sp(2)\times Sp(2)}$

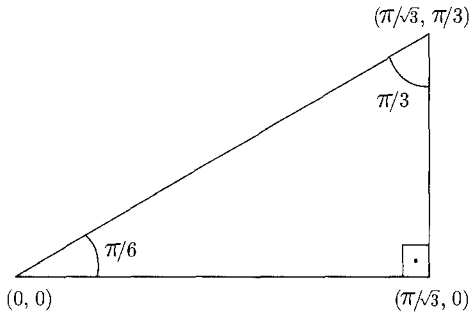

Figure 3 – Orbit space in the case $\circ \equiv \circ$

j	a_{α_j}	b_{α_j}	m_{α_j}				
			$\frac{Sp(4)}{Sp(2)\times Sp(2)}$	$\frac{SU(2+m)}{S(U(2)\times U(m))}$ $m>2$	$\frac{Sp(2+m)}{Sp(2)\times Sp(m)}$ $m>2$	$\frac{SO(10)}{U(5)}$	$\frac{E_6}{U(1)\times_{\mathbb{Z}_2} Spin(10)}$
1	0	1	4	$2(m-2)$	$4(m-2)$	4	8
2	1	-1	3	2	4	4	6
3	1	1	3^*	2	4	4	6
4	1	0	4	$2(m-2)$	$4(m-2)$	4	8
5	0	2	–	1	3	1	1
6	2	0	–	1^*	3^*	1^*	1^*

Table 1: Restricted roots for the case $\circ = \circ$, $d = 4$.

j	a_{α_j}	b_{α_j}	m_{α_j}	
			G_2	$G_2/SO(4)$
1	0	1	2	1
2	$\sqrt{3}/2$	$-3/2$	2	1
3	$\sqrt{3}/2$	$-1/2$	2	1
4	$\sqrt{3}/2$	$1/2$	2	1
5	$\sqrt{3}/2$	$3/2$	2	1
6	$\sqrt{3}$	0	2^*	1^*

Table 2: Restricted roots for the case $\circ \equiv \circ$, $d = 6$.

Observe that the vertex $(0,0)$ of X is a fixed point for the K-action. We wish to develop a method to construct a non-self-intersecting solution of eqn. (2) which starts at the boundary $y=0$ of X and terminates at the boundary $y = x\tan\pi/d$. In the next sections we will describe an approximation technique suitable for findind the desired solution and then we will apply it to the case of G_2.

4. Comparison Lemma

We want to consider solutions to eqn. (2) with initial condition $y(0)=0$. Then, $\sigma(0)=\pi/2$. We define $v=-\cos\sigma$ and, as long as $\sin\sigma>0$, we may rewrite eqn. (2) in the following form in which y is an independent variable:

$$\begin{aligned} \frac{dx}{dy} &= \frac{-v}{\sqrt{1-v^2}} \\ \frac{dv}{dy} &= W(x,y,v) \end{aligned} \tag{3}$$

where

$$W(x,y,v) = -\sum_{j=1}^{6} m_{\alpha_j}\cot(a_{\alpha_j}x + b_{\alpha_j}y)(b_{\alpha_j}v + a_{\alpha_j}\sqrt{1-v^2}).$$

The comparison lemma below is taken from [**10**]. It guarantees that an actual solution of eqn. (3) is *sandwiched* between two approximating curves up to first order. The key ingredient in its proof is the fact that the function $x\mapsto W(x,y,v)$ is increasing for y fixed and for $v\in[0,\cos(\pi/2+\pi/d)]$ fixed.

Lemma 1. *Let $y\mapsto(x(y),v(y))$ be the solution of eqn. (3) with initial conditions $x(y_0)=x_0$, $v(y_0)=v_0$ ($v_0=0$ if $y_0=0$). Let $x_j=x_j(y)$, $j:1,2$, satisfy $x_j(y_0)=x_0$, $v_j(y_0)=v_0$ where $v_j(y) = -x_j'(y)/\sqrt{1+x_j'(y)^2}$, $j:1,2$, and suppose*

$$x_1(y)<x_2(y) \quad \textit{and} \quad \cos\pi/d > v_1(y) > v_2(y) > 0 \qquad \textit{for } y\in(y_0,y_1].$$

Define

$$w_j(y) = W(x_{3-j}(y), y, v_j(y)), \qquad \textit{for } j:1,2.$$

Assume further that

$$v_1'(y) > v'(y) > v_2'(y) \qquad \textit{for } y - y_0 > 0 \textit{ small,}$$

and

$$v_1'(y) > w_1(y), \quad w_2(y) > v_2'(y) \qquad \textit{for } y \in (y_0, y_1].$$

Then

$$x_1(y) < x(y) < x_2(y) \quad \textit{and} \quad v_1(y) > v(y) > v_2(y) \qquad \textit{for } y \in (y_0, y_1].$$

Proof. Since $v_1'(y) > v'(y) > v_2'(y)$ for small $y - y_0 > 0$ and $v_1(y_0) = v_2(y_0) = v(y_0) = v_0$, $x_1(y_0) = x_2(y_0) = x(y_0) = x_0$, we have $v_1(y) > v(y) > v_2(y)$ and $x_1(y) < x(y) < x_2(y)$ for $y - y_0 > 0$ small. Without loss of generality, suppose by contradiction there is an $y_2 \in (y_0, y_1]$ such that $x(y_2) = x_2(y_2)$. Then there must be an $y_3 \in (y_0, y_2)$ such that $v(y_3) = v_2(y_3)$. Let $\bar{y}$ be the first $\bar{y} > 0$ such that $v(\bar{y}) = v_2(\bar{y})$. Then $v'(\bar{y}) \leq v_2'(\bar{y})$. We may also assume that $x(\bar{y}) > x_1(\bar{y})$. Now we have

$$\begin{aligned} v'(\bar{y}) &= W(x(\bar{y}), \bar{y}, v(\bar{y})) \\ &> W(x_1(\bar{y}), \bar{y}, v_2(\bar{y})) \\ &= w_2(\bar{y}) \end{aligned}$$

Thus, $v_2'(\bar{y}) > w_2(\bar{y})$, a contradiction. □

For the sake of future reference, we now state the comparison lemma for solutions starting at the boundary $y = x \tan \pi/d$. Let us change to the coordinates $(z, w) = (x \cos \pi/d + y \sin \pi/d, x \sin \pi/d - y \cos \pi/d)$ and put $\tau = -\sigma + \pi/d$, $u = -\cos \tau$. Then eqn. (3) can be rewritten

$$\begin{aligned} \frac{dz}{dw} &= \frac{-u}{\sqrt{1-u^2}} \\ \frac{du}{dw} &= Y(z, w, u) \end{aligned} \tag{4}$$

where

$$Y(z,w,u) = -\sum_{j=1}^{6} m_{\alpha_j} \cot[(a_{\alpha_j} \cos\pi/d + b_{\alpha_j} \sin\pi/d)z+ \\ + (a_{\alpha_j} \sin\pi/d - b_{\alpha_j} \cos\pi/d)w]\times \\ \times [(-b_{\alpha_j} \cos\pi/d + \alpha_{\alpha_j} \sin\pi/d)u+ \\ + (b_{\alpha_j}\ sin\pi/d + a_{\alpha_j} \cos\pi/d)\sqrt{1-u^2}]$$

Lemma 2. *Let $w \mapsto (z(w), u(w))$ be the solution of eqn. (4) with initial conditions $z(w_0) = z_0$, $u(w_0) = u_0$ ($u_0 = 0$ if $w_0 = 0$). Let $z_j = z_j(w)$, $j : 1, 2$, satisfy $z_j(w_0) = z_0$, $u_j(w_0) = u_0$ where*

$$u_j(w) = -z_j'(w)/\sqrt{1+z_j'(w)^2}, \quad j:1,2,$$

and suppose

$$z_1(w) < z_2(w) \quad \textit{and} \quad \cos\pi/d > u_1(w) > u_2(w) > 0$$

for $w \in (w_0, w_1]$. Define

$$y_j(w) = Y(z_{3-j}(w), w, u_j(w)), \qquad \textit{for } j : 1,2.$$

Assume further that

$$u_1'(w) > u'(w) > u_2'(w) \qquad \textit{for } w - w_0 > 0 \textit{ small,}$$

and

$$u_1'(w) > y_1(w), \quad y_2(w) > u_2'(w) \qquad \textit{for } w \in (w_0, w_1].$$

Then

$$z_1(w) < z(w) < z_2(w) \quad \textit{and} \quad u_1(w) > u(w) > u_2(w)$$

for $w \in (w_0, w_1]$.

5. Foliations by Solution Curves

We want to establish a corridor starting at the boundary $y = 0$ such that the family of solutions of eqn. (3) with initial point in a certain subinterval of that boundary will be guaranteed to comprise a foliation of the orbit space X, up to a certain height. This will be achieved by linearizing eqn. (3) with respect to x_0, where $(x_0, 0)$ is the initial condition of a fixed solution curve $y \mapsto (x(y), v(y))$ defined for $y \in [0, y_1]$.

The linearization is

$$\begin{aligned} \bar{x}' &= A(y)\bar{v} \\ \bar{v}' &= B(y)\bar{x} + C(y)\bar{v} \end{aligned} \tag{5}$$

where the initial conditions are $\bar{x}(0) = 1$, $\bar{v}(0) = 0$ and

$$\begin{aligned} A(y) &= -(1 - v(y)^2)^{-3/2}, \\ B(y) &= \sum_{j=2}^{6} m_{\alpha_j} a_{\alpha_j} \csc^2(a_{\alpha_j} x(y) + b_{\alpha_j} y)(b_{\alpha_j} v + a_{\alpha_j} \sqrt{1 - v^2}), \\ C(y) &= -\sum_{j=1}^{6} m_{\alpha_j} \cot(a_{\alpha_j} x(y) + b_{\alpha_j} y)(b_{\alpha_j} - a_{\alpha_j} \frac{v}{\sqrt{1 - v^2}}), \end{aligned}$$

are functions defined along the solution curve $y \mapsto (x(y), v(y))$. Now eqn. (5) is equivalent to the following

$$[p(y)\bar{x}'(y)]' + q(y)\bar{x}(y) = 0, \tag{6}$$

where

$$\begin{aligned} p(y) &= e^{\int^y f_1}, \\ q(y) &= f_2(y)p(y), \end{aligned}$$

and

$$\begin{aligned} f_1(y) &= -\frac{A'(y)}{A(y)} - C(y), \\ f_2(y) &= -A(y)B(y). \end{aligned}$$

Let $\bar{x}(y)$ be a solution of eqn. (6) satisfying the initial conditions $\bar{x}(0) = 1$, $\bar{x}'(0) = 0$. We know that if $\bar{x}(y) > 0$ for $y \in (0, y_1]$ and $x_0 \in [a, b]$, then $(x_0, y) \in [a, b] \times [0, y_1] \mapsto (x(y), y)$ defines an analytic foliation of its image. So we need to estimate the zeros of the solution for the initial value problem for $\bar{x}(y)$, and in order to do that we apply the standart Sturm comparison method to eqn. (6).

Lemma 3. **(Sturm)** *Let $\psi(y)$ be a solution of eqn. (6) defined for $y \in [0, y_1]$ with $\psi(0) = 1$, $\psi'(0) = 0$. Let $\tilde{q}(y) > q(y)$ for $y \in (0, y_1]$ and suppose that $\tilde{\psi}(y)$ is the solution of*

$$[p(y)\tilde{\psi}'(y)]' + \tilde{q}(y)\tilde{\psi}(y) = 0 \tag{7}$$

with $\tilde{\psi}(0) = 1$, $\tilde{\psi}'(0) = 0$. *If* $\tilde{\psi}(y)$ *never vanishes on* $[0, y_1]$, *then* $\psi(y)$ *never vanishes on* $[0, y_1]$.

Proof. Suppose that y_2 is the first zero of ψ in $[0, y_1]$. Then, since $\tilde{\psi}(y) > 0$ on $[0, y_2]$, we get:

$$(p\psi')'\tilde{\psi} - (p\tilde{\psi}')'\psi = (\tilde{q} - q)\psi\tilde{\psi} > 0$$

on $(0, y_2)$; integrating, we obtain

$$\int_0^{y_2} [(p\psi')'\tilde{\psi} - (p\tilde{\psi}')'\psi] dy > 0.$$

Now, since

$$(p\psi'\tilde{\psi})' - (p\tilde{\psi}'\psi)' = (p\psi')'\tilde{\psi} - (p\tilde{\psi}')'\psi,$$

and using the various initial conditions, it follows that

$$p(y_2)\psi'(y_2)\tilde{\psi}(y_2) > 0.$$

But this contradicts the fact that $p(y_2) > 0$, $\tilde{\psi}(y_2) > 0$ and $\psi'(y_2) \leq 0$. This last inequality follows from the assumption that y_2 is the first zero of ψ. $\square$

Remark. *In the case we are dealing with,* $p(0) = 0$, *so that the requirement that* $\psi'(0) = \tilde{\psi}'(0) = 0$ *is not necessary. For example, the following application of Lemma 3 will suffice.*

Lemma 4. *Let* $\psi(y)$ *be a solution of eqn. (6) defined for* $y \in [0, y_1]$ *and satisfying the initial conditions* $\psi(0) = 1$, $\psi'(0) = 1$. *If*

$$\inf_{y \in (0, y_1]} \frac{-f_2(y)}{f_1(y) + y f_2(y)} > -\frac{1}{y_1}$$

then $\psi(y)$ *never vanishes on* $[0, y_1]$.

Proof. In fact, we may choose

$$\inf_{y \in (0, y_1]} \frac{-f_2(y)}{f_1(y) + y f_2(y)} > k > -\frac{1}{y_1}$$

and define $\tilde{\psi}(y) = 1 + ky$. Then the above inequalities imply that $\tilde{y} > 0$ on $[0, y_1]$ and $\tilde{\psi}$ satisfies eqn. (7) where

$$\tilde{q}(y) = \frac{-kp'(y)}{1 + ky} > q(y)$$

for $y \in (0, y_1]$. Now use Remark 1 and apply Lemma 3. $\square$

6. Polynomial Approximations

We know that the solution curves of eqn. (3) are analytic. Therefore, in order to find the approximations x_1, x_2 to a particular solution as described in Section 4, it is reasonable to use the power series expansion of that solution.

We recall eqn. (3):

$$\begin{aligned} \frac{dx}{dy} &= \frac{-v}{\sqrt{1-v^2}} \\ \frac{dv}{dy} &= W(x,y,v) \end{aligned} \tag{8}$$

where

$$W(x,y,v) = -\sum_{j=1}^{6} m_{\alpha_j} \cot(a_{\alpha_j} x + b_{\alpha_j} y)(b_{\alpha_j} v + a_{\alpha_j} \sqrt{1-v^2}).$$

It is easy to compute the Taylor expansion of a solution $y \mapsto (x(y), v(y))$ at $y = 0$, namely

$$\begin{aligned} x(y) &= \sum_{i=0}^{\infty} \frac{x_i}{(2i)!} y^{2i}, \\ v(y) &= \sum_{i=0}^{\infty} \frac{v_i}{(2i+1)!} y^{2i+1}. \end{aligned}$$

Let $m = m_{\alpha_1}$ and

$$R(y) = -\sum_{j=2}^{6} m_{\alpha_j} \cot(a_{\alpha_j} x(y) + b_{\alpha_j} y)(b_{\alpha_j} v(y) + a_{\alpha_j} \sqrt{1-v(y)^2}).$$

Then eqn. (8) is

$$\begin{aligned} \frac{dx}{dy} &= \frac{-v}{\sqrt{1-v^2}} \\ \frac{dv}{dy} &= R(y) - mv \cot y \end{aligned}$$

Here we specialize to the case $\circ \equiv \circ$. Then R is exactly the "non-singular" part, and so the coefficients of the Taylor expansion can be inductively computed as

$$v_0 = \frac{R(0)}{1+m},$$

$$v_n = \frac{2n+1}{2n+m+1}\left[m(2n)!\sum_{i=0}^{n-1}\frac{v_i}{(2i+1)!}\frac{B_{2(n-i)}}{(2(n-i))!}2^{2(n-i)} + R^{(2n)}(0)\right],$$

$n \geq 1$, where B_i is the ith Bernoulli number, and

$$\begin{aligned}
x_1 &= -v_0,\\
x_2 &= -3v_0^3 - v_1,\\
x_3 &= -45v_0^5 - 30v_0^2 v_1 - v_2,\\
x_4 &= -1575v_0^7 - 1575v_0^4 v_1 - 210 v_0 v_1^2 - 63 v_0^2 v_2 - v_3,\\
x_5 &= -99225 v_0^9 - 132300 v_0^6 v_1 - 37800 v_0^3 v_1 - 840 v_1^3 - 5670 v_0^4 v_2 -\\
&\qquad - 1512 v_0 v_1 v_2 - 108 v_0^2 v_3 - v_4, \quad \text{etc.}
\end{aligned}$$

For solutions starting at the top boundary, the Taylor expansion at $w = 0$ of a solution $w \mapsto (z(w), u(w))$ to eqn. (4) is

$$z(w) = \sum_{i=0}^{\infty} \frac{z_i}{(2i)!} w^{2i},$$

$$u(w) = \sum_{i=0}^{\infty} \frac{u_i}{(2i+1)!} w^{2i+1}.$$

Let $m = m_{\alpha_2}$, $a = a_{\alpha_2}$ and

$$\begin{aligned}
R(w) = -\sum_{\substack{j=1\\ j\neq 2}}^{6} m_{\alpha_j} \cot[(a_{\alpha_j}\cos\pi/d + b_{\alpha_j}\sin\pi/d)z(w) +&\\
+ (a_{\alpha_j}\sin\pi/d - b_{\alpha_j}\cos\pi/d)w] \times&\\
\times [(-b_{\alpha_j}\cos\pi/d + a_{\alpha_j}\sin\pi/d)u(w) +&\\
(b_{\alpha_j}\sin\pi/d + a_{\alpha_j}\cos\pi/d)\sqrt{1-u(w)^2}].&
\end{aligned}$$

Then eqn. (4) is

$$\frac{dz}{dw} = \frac{-u}{\sqrt{1-u^2}}$$

$$\frac{du}{dw} = R(w) - mau\cot(aw)$$

and the coefficients can be computed inductively as

$$u_0 = \frac{R(0)}{1+m},$$
$$u_n = \frac{2n+1}{2n+m+1}\left[m(2n)!\sum_{i=0}^{n-1}\frac{u_i}{(2i+1)!}\frac{B_{2(n-i)}}{(2(n-i))!}(2a)^{2(n-i)} + R^{(2n)}(0)\right],$$

$n \geq 1$, where B_i is the ith Bernoulli number, and

$$z_1 = -u_0,$$
$$z_2 = -3u_0^3 - u_1,$$
$$z_3 = -45u_0^5 - 30u_0^2u_1 - u_2,$$
$$z_4 = -1575u_0^7 - 1575u_0^4u_1 - 210u_0u_1^2 - 63u_0^2u_2 - u_3,$$
$$z_5 = -99225u_0^9 - 132300u_0^6u_1 - 37800u_0^3u_1 - 840u_1^3 - 5670u_0^4u_2 - \\ - 1512u_0u_1u_2 - 108u_0^2u_3 - u_4, \quad \text{etc.}$$

7. An Example: the Case of G_2

Now we show how the above ideas can be applied to construct a solution to eqn. (2) starting at the bottom boundary $y = 0$ and terminating at the top boundary $y = x\tan\pi/d$; in fact, a solution to eqn. (3). Since the theoretical bases of such construction are the same for all cases involved, we shall work out the details in the specific case of G_2.

Denote by $y \mapsto (X_{x_0}(y), V_{x_0}(y))$ the solution of eqn. (3) with initial condition $(x_0, 0)$ at $y = 0$. Denote by $w \mapsto (Z_{z_0}(w), U_{z_0}(w))$ the solution of eqn. (4) with initial condition $(z_0, 0)$ at $w = 0$.

First step: *use the comparison lemma to find approximations for solutions starting at the bottom boundary.*

Let $x_0 \in [1.46, 1.47]$ and define

$$X_{1,x_0}(y) = x_0 + \frac{x_1}{2}y^2 + \frac{x_2}{24}y^4 + (8.65 - 6x_0)y^6,$$
$$X_{2,x_0}(y) = x_0 + \frac{x_1}{2}y^2 + \frac{x_2}{24}y^4 + (8.76 - 6x_0)y^6$$

where x_1, x_2 are coefficients of the Taylor expansion at $y = 0$ of the solution $X_{x_0}(y)$ with $X_{x_0}(0) = x_0$ (see Section 6). Then use the computer to verify the last condition of Lemma 1 with $y_1 = 0.41$ (the other conditions are very simple to verify), that is, if

$$V_{j,x_0}(y) = -X'_{j,x_0}(y)/\sqrt{1+X'_{j,x_0}(y)^2} \qquad \text{and}$$
$$W_{j,x_0}(y) = W(X_{3-j,x_0}(y), y, V_{y,x_0}(y))$$

for $j : 1, 2$, then

$$V'_{1,x_0}(y) > W_{1,x_0}(y) \quad \text{and} \quad W_{2,x_0}(y) > V'_{2,x_0}(y) \qquad \text{for } y \in (0, 0.41].$$

We conclude that X_{1,x_0}, X_{2,x_0} provide $\mathcal{C}^1$ bounds for the actual solution X_{x_0} which are valid on the interval $[0, 0.41]$, that is,

$$X_{1,x_0}(y) < X_{x_0}(y) < X_{2,x_0}(y), \quad V_{1,x_0}(y) > V_{x_0}(y) > V_{2,x_0}(y)$$

for $y \in (0, 0.41]$.

Second step: *use the above $\mathcal{C}^1$ bounds establish a corridor foliated by solutions starting at the bottom boundary.*

The $\mathcal{C}^1$ bounds X_{1,x_0} and X_{2,x_0} for the actual solution X_{x_0} enable us to estimate (with the aid of a computer) that

$$\inf_{y\in(0,0.36]} \frac{-f_2(y)}{f_1(y) + yf_2(y)} > -2.72,$$
$$-\frac{1}{0.36} < -2.72,$$

for $x_0 \in [1.46, 1.47]$. Then we apply Lemma 4 and it follows from the argument of Section 5 that the $X_{x_0}(y)$ for $x_0 \in [1.46, 1.47]$ foliate a region of the orbit space from $y = 0$ until $y = 0.36$.

Third step: *use the comparison lemma to find approximations for solutions starting at the top boundary.*

Let $z_0 \in [1.4202816622, 1.4549226784]$ and define

$$Z_{1,z_0}(w) = z_0 + \frac{z_1}{2}w^2 + \frac{z_2}{24}w^4 + \frac{z_3}{720}w^6 + \frac{z_4}{40320}w^8 + (212.3499997012 - (149.3893819431)z_0)w^{10},$$
$$Z_{2,z_0}(w) = z_0 + \frac{z_1}{2}w^2 + \frac{z_2}{24}w^4 + \frac{z_3}{720}w^6 + \frac{z_4}{40320}w^8 + (71.9789999008 - (49.6232555672)z_0)w^{10},$$

where z_1, z_2, z_3 and z_4 are coefficients of the Taylor expansion at $w = 0$ of the solution $Z_{z_0}(w)$ with $Z_{z_0}(0) = z_0$ (see Section 6). Then proceed in analogy with the first step and apply Lemma 2 with $w_1 = 0.55$ to conclude that Z_{1,z_0}, Z_{2,z_0} provide $\mathcal{C}^1$ bounds for the actual solution $Z_{z_0}(w)$ which are valid on the interval $[0, 0.55]$.

Fourth step: *study the intersections of solutions starting at the top boundary with solutions starting at the bottom boundary.*

Let $X(y)$, $Z(w)$ be solutions to eqns. (3) and (4), respectively, starting at the bottom and top boundaries, respectively, and consider the corresponding $\mathcal{C}^1$ approximations $X_1(y)$, $X_2(y)$ and $Z_1(w)$, $Z_2(w)$, respectively. In order to show that $X(y)$ and $Z(w)$ cross each other, we consider the following function:

$$\mathcal{F} : y \in [y_a, y_b] \mapsto Z(T_2(X(y), y)) - T_1(X(y), y),$$

where

$$T(x, y) = (T_1(x, y), T_2(x, y)) = (\frac{1}{2}(x\sqrt{3} + y), \frac{1}{2}(x - y\sqrt{3}))$$

is the change from the coordinates (z, w) to the coordinates (x, y). We want to show that it has a zero. For that, it is enough to show that it assumes a positive value and a negative value on $[y_a, y_b]$. Let $(z_a^2, w_a^2) = T(X_2(y_a), y_a)$, $(z_a, w_a) = T(X(y_a), y_a)$. Since $X(y_a) < X_2(y_a)$, we have $z_a < z_a^2$. Also $Z_1(w_a) < Z(w_a)$. Therefore,

$$z_a^2 < Z_1(w_a) \Rightarrow z_a < Z(w_a) \Rightarrow \mathcal{F}(y_a) > 0$$

Let $(z_b^1, w_b^1) = T(X_1(y_b), y_b)$, $(z_b, w_b) = T(X(y_b), y_b)$. Since $X_1(y_b) < X(y_b)$, we have $z_b^1 < z_b$. Also $Z(w_b) < Z_2(w_b)$. Therefore,

$$Z_2(w_b) < z_b^1 \Rightarrow Z(w_b) < z_b \Rightarrow \mathcal{F}(y_b) < 0$$

Thus, the conditions for $X(y)$ and $Z(w)$ to cross each other on $[y_a, y_b]$ at an angle (measured from the X-direction to the Z-direction; see Fig. 4) less than π are

$$z_a^2 < Z_1(w_a) \quad \text{and} \quad Z_2(w_b) < z_b^1.$$

Since X_1, X_2, Z_1, Z_2 are decreasing, those conditions are satisfied if

$$T_1(X_2(y_a), y_a) < Z_1(T_2(X_2(y_a), y_a))$$

and

$$Z_2(T_2(X_1(y_b), y_b)) < T_1(X_1(y_b), y_b).$$

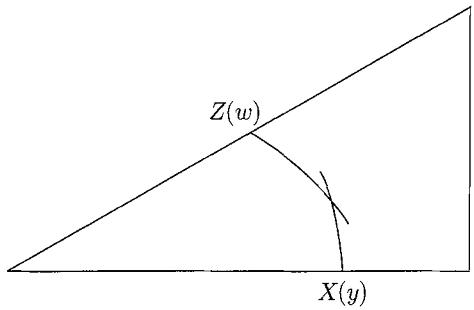

Figure 4 – Crossing at an angle $< \pi$

Similarly, the conditions for X and Z to cross each other on $[y_a, y_b]$ at an angle (measured from the X-direction to the Z-direction; see Fig. 5) more than π are

$$Z_2(T_2(X_1(y_a), y_a)) < T_1(X_1(y_a), y_a)$$

and

$$T_1(X_2(y_b), y_b) < Z_1(T_2(X_2(y_b), y_b)).$$

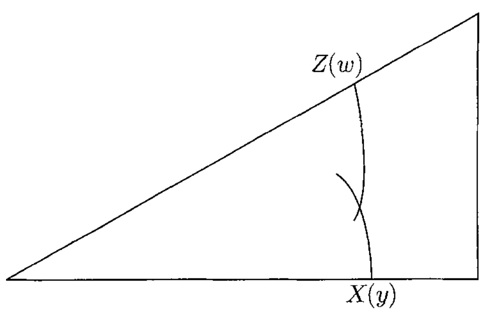

Figure 5 - Crossing at an angle $> \pi$

From the first step, we have the following $\mathcal{C}^1$ approximations valid for $y \in [0, 0.41]$ for the solution $X_{x_0}(y)$ starting at the bottom boundary with $X_{x_0}(0) = x_0$.

If $x_0 = 1.46$, then

$$X_{1,1.46}(y) = 1.46 - 0.4558551314y^2 + 0.1447836669y^4 - 0.11y^6,$$
$$X_{2,1.46}(y) = 1.46 - 0.4558551314y^2 + 0.1447836669y^4 - 0.y^6.$$

If $x_0 = 1.47$, then

$$X_{1,1.47}(y) = 1.47 - 0.4978218951y^2 + 0.1562579343y^4 - 0.17y^6,$$
$$X_{2,1.47}(y) = 1.47 - 0.4978218951y^2 + 0.1562579343y^4 - 0.06y^6.$$

From the third step, we have the following $\mathcal{C}^1$ approximations valid for $w \in [0, 0.55]$ for the solution $Z_{z_0}(w)$ starting at the top boundary with $Z_{z_0}(0) = z_0$.

If $z_0 = 1.4202816622$, then

$$Z_{1,z_0}(w) = 1.4202816622 - 0.1881760127w^2 - 0.0834483267w^4 + \\ + 0.01540925912w^6 - 0.0081428042w^8 + 0.175w^{10},$$
$$Z_{2,z_0}(w) = 1.4202816622 - 0.1881760127w^2 - 0.0834483267w^4 + \\ + 0.01540925912w^6 - 0.0081428042w^8 + 1.5w^{10}$$

(this one is in fact valid for $w \in (0, 0.58]$).

If $z_0 = 1.4549226784$, then

$$Z_{1,z_0}(w) = 1.4549226784 - 0.2881043431w^2 - 0.1812941968w^4 - \\ - 0.09184112229w^6 - 0.05990390855w^8 - 5.w^{10},$$
$$Z_{2,z_0}(w) = 1.4549226784 - 0.2881043431w^2 - 0.1812941968w^4 - \\ - 0.09184112229w^6 - 0.05990390855w^8 - 0.219w^{10}.$$

Consider the foliated corridor

$$(x_0, y) \in [1.46, 1.47] \times [0, 0.28] \mapsto X_{x_0}(y).$$

Then $Z_{z_0}(w)$ enters the foliated corridor at $y = 0.28$ for all $z_0 \in [1.42\ldots, \ldots 1.45\ldots]$. In fact, the segment $\{(x, y = 0.28) : x \in [X_{2,1.46}(0.28) = 1.4251508792, X_{1,1.47}(0.28) = 1.4318492908]\}$ is described in (z, w)-coordinates by $z = w\sqrt{3} + 0.56$, $w \in [0.4700883265, 0.4734375323]$, and we may check that given $z_0 \in [1.42\ldots, 1.45\ldots]$, the function $Z_{j,z_0}(w) - (w\sqrt{3} + 0.56)$ changes sign for $w \in [0.470\ldots, 0.473\ldots]$.

Next it is easy to apply the method given above to check that $Z_{1.42\ldots}(w)$ crosses $X_{1.46}(y)$ for some $y \in [0.28, 0.30]$ and that it crosses

$X_{1.47}(y)$ for some $y \in [0.20, 0.23]$ (in that range, $w < 0.56$, so that the approximations are still valid). Also, we check that $Z_{1.45\ldots}(w)$ crosses $X_{1.47}(y)$ for some $y \in [0.28, 0.31]$ and that it crosses $X_{1.46}(y)$ for some $y \in [0.20, 0.25]$ (in that range, $w < 0.55$, so that the approximations are still valid).

Now consider the extended foliated corridor $(x_0, y) \in [1.46, 1.47] \times [0, 0.36] \mapsto X_{x_0}(y)$. Since $Z_{1.42\ldots}(w)$ enters this longer foliated corridor by crossing $X_{1.46}(y)$ at an angle less than π, $Z_{1.42\ldots}(w)$ must remain inside this corridor and cross every solution $X_{x_0}(y)$, $x_0 \in (1.46, 1.47)$, at an angle less than π until it reaches the crossing with $X_{1.47}(y)$ (at an angle less than π) and exits the corridor. Similarly, the solution $Z_{1.45\ldots}(w)$ enters the longer foliated corridor by crossing $X_{1.47}(y)$ at an angle more than π, and so, it must remain inside this corridor and cross every solution $X_{x_0}(y)$, for x_0 from 1.47 to 1.46, at an angle more than π.

In particular, $Z_{1.42\ldots}(w)$ enters the shorter foliated corridor $(x_0, y) \in [1.46, 1.47] \times [0, 0.28] \mapsto X_{x_0}(y)$ with an angle less than π and $Z_{1.45\ldots}(w)$ enters the shorter foliated corridor with an angle more than π. By continuity there must be an $x_0 \in (1.46, 1.47)$ and a $z_0 \in (1.42\ldots, 1.45\ldots)$ such that $Z_{z_0}(w)$ enters the shorter foliated corridor intersecting $X_{x_0}(y)$ with an angle equal to π, that is, $Z_{z_0}(w)$ and $X_{x_0}(y)$ are "the same" curve (see Fig. 6). Thus, we get a solution connecting the top and bottom boundaries and we have proved

Theorem 1. *There is an embedded minimal hypersphere in the exceptional Lie group G_2.*

A similar argument will prove the existence of an embedded minimal hypersphere in each one of

$$Sp(4)/(Sp(2) \times Sp(2)), G_2/SO(4), SU(5)/S(U(2) \times U(3)),$$
$$Sp(5)/(Sp(2) \times Sp(3)), SO(10)/U(5), \quad \text{and} \quad E_6/(U(1) \times_{\mathbb{Z}_2} Spin(10)).$$

In the first two cases the generating solution curve of eqn. (2) connects the boundary $y = x \tan \pi/d$ to the boundary $y = 0$, as in the case of G_2. However, in the remainig four cases ($d = 4$), the vertex $(\pi/2, \pi/2)$ ia also a fixed point of the corresponding K-action and the generating solution curve of eqn. (2) connects the boundary $y = x$ to the boundary $x = \pi/2$.

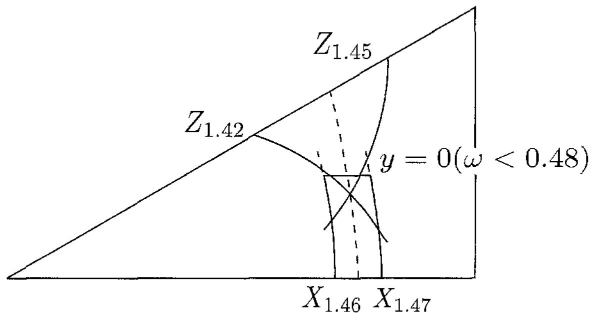

Figure 6 - Auxiliary solutions entering the foliated corridor

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